Nature of singularities in gravitational collapse

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We discuss several aspects of cosmic censorship hypothesis. There is evidence both in favor of and against the hypothesis. On one hand one can prove that cosmic censorship holds in several special cases and on the other hand there are a number of special solutions of Einstein equations in which it is violated. One way to resolve cosmic censorship problem is to test it observationally. We point out several possibilities of such tests using present and future instruments.

§1. Introduction

In a recent significant publication on the subject of space-time singularities we read.

"*One of the fundamental unanswered questions in the general theory of relativity is whether* **naked** *singularities, that is singular events which are visible from infinity, may form with positive probability in the process of gravitational collapse. The conjecture that the answer to this question is in negative has been called* **cosmic censorship**" [1]. **

This clearly indicates that the fundamental question posed by Roger Penrose [2] that is cosmic censorship hypothesis remains unresolved. In this work I shall discuss some aspects of the cosmic censorship hypothesis that are close to my own work on this problem. Moreover I shall point out possible observational tests of cosmic censorship. Other aspects of this conjecture are presented in a number of contributions to these proceedings. Excellent exposition of the cosmic censorship problem can be found in recent reviews of Chruściel [3], Clarke [4], Wald [5], Hawking and Penrose [6], Joshi [7], Singh [8].

My review will be divided into three parts. First I shall present some theorems using methods of global Lorentzian geometry restricting the occurrence of naked singularities, next I shall discuss examples of naked singularities in solutions of Einstein equations, and finally I shall point out a number of possibilities of observational verification of cosmic censorship.

§2. Cosmic censorship: geometrical approach

In 1969 [2] Roger Penrose put forward a hypothesis that there exists a "**cosmic censor**" that forbids occurrence of "naked singularities".

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** It is interesting that physical concepts like "gravitational collapse" and physicists jargon like "naked singularities" and "cosmic censorship" are getting into pure mathematics literature.

In the language of Lorentzian geometry naked singularity is a *timelike ideal point* of the boundary of space-time. The basic definitions and concepts of Lorentzian geometry can be found in the monograph of Hawking and Ellis [15)].

Definition 1 *A future-directed (past-directed) causal curve λ terminates in a timelike ideal point if there exists a point p of space-time such that the chronological past $I^-(\lambda) \subset I^-(p)$ (respectively chronological future $I^+(\lambda) \subset I^+(p)$).*

Penrose showed [9)] that the absence of timelike ideal points is equivalent to global hyperbolicity of space-time.

There are two versions of cosmic censorship: a strong one and a weak one. The strong version says that space-time is globally hyperbolic. The weak version says that the intersection of the causal future of a partial Cauchy surface and the causal past of the boundary at infinity is globally hyperbolic. Thus weak cosmic censorship says that there is no naked singularity outside the black hole event horizons. For simplicity we shall not discuss possibility of violation of global hyperbolicity at infinity. For example in the Oppenheimer-Snyder model [10)] of gravitational collapse the strong cosmic censorship holds whereas for Kerr space-time with $a^2 < m^2$ only the weak one is valid as the singularity that is hidden behind the event horizon is timelike. Cosmic censorship does not say that there is no singularity visible to observers. Clearly initial cosmological singularity is visible to all observers. Rather it says that there exists an initial surface from which we can predict evolution of the whole of space-time (strong version) or part of the space-time outside black holes (weak version).

When space-time is not globally hyperbolic there is a partial Cauchy surface S such that Cauchy horizon $H^+(S)$ is not empty. Thus study of cosmic censorship can be reduced to the study of the existence of Cauchy horizons. A Cauchy horizon is a Lipschitz (C^{1-}) 3-dimensional manifold. This, in turn, implies that Cauchy horizons are differentiable almost everywhere. Because they are differentiable except for a set of (three-dimensional) measure zero, it seems that they have often been assumed to be smooth except for a set that may be more or less neglected. However, one must remember in the above that: (1) the term "differentiable" refers to being differentiable at a single point, and (2) sets of measure zero may be quite widely distributed. In fact Chruściel and Galloway [11)] have constructed examples of nowhere differentiable Cauchy horizons and Budzyński, Kondracki and this author have shown [12)] that a class of nowhere differentiable Cauchy horizons is large.

We shall say that a Cauchy horizon H is **smooth** if it *contains an open set G where its is C^2 and such that complement of G in H has measure zero.* Throughout the rest of this paper we shall assume that all Cauchy horizons are smooth.

The geometrical techniques to study the large-scale structure of space-time developed by Geroch, Hawking, and Penrose use extensively an ordinary non-linear equation of Riccati type known as Raychaudhuri-Newman-Penrose (RNP) equation. Let t be an affine parameter on a null geodesic λ, K^a be components of the tangent vector to λ, θ be expansion, σ be shear and R_{ab} be components of the Ricci tensor. then the RNP equation takes the form.

$$\frac{d\theta}{dt} = -\frac{1}{2}\theta^2 - 2\sigma^2 - R_{ab}K^aK^b, \tag{2·1}$$

The quantity θ describes the expansion of congruences of null geodesics infinitesimally neighboring λ and it is defined as $\theta = \frac{1}{A}\frac{dA}{dt}$ where A is cross-section of the congruence.

The great success of these techniques was the proof of the existence of singularities (defined as incomplete causal geodesics) in general space-times [13]. The techniques can be also used to study the problem of cosmic censorship. We shall next review the results of this and other authors in that direction. We shall proceed as follows. We introduce a series of conditions on space-time and we discuss how each of them restricts the occurrence of naked singularities.

Condition 1 (Null convergence condition) *We say that the null convergence condition holds if* $R_{ab}K^aK^b \geq 0$ *for all null vectors* K.

By Einstein equations this condition is satisfied by all reasonable classical matter models.

Definition 2 *Let* S *be a partial Cauchy surface. A future Cauchy horizon* $H^+(S)$ *is compactly generated if all its generators, when followed their past, enter and remain in a compact subset* C.

The above class of Cauchy horizons has been introduced by Hawking [14] to describe a situation in which a Cauchy horizon arises as a result of causality violation rather than singularities or timelike boundary at infinity.

Theorem 1 (Hawking 1992 [14]) *If null convergence condition holds then a compactly generated Cauchy horizon that is non-compact cannot arise.*

Thus under a very mild - from physical point of view - restriction on space-time a nontrivial class of Cauchy horizons is ruled out.

Let R_{abcd} be components of Riemann tensor. We say that an endless null geodesic γ is **generic** if for some point p on γ $\;K^cK^dK_{[a}R_{b]cd[e}K_{f]} \neq 0$ where K is a vector tangent to γ at p.

Condition 2 (Generic condition) *All endless null geodesics in space-time are generic.*

This condition means that every null geodesic encounters some curvature that is not specially aligned with geodesic.

Theorem 2 *If null convergence condition holds and one of the null geodesic generators of a Cauchy horizon* H *is generic then* H *cannot be compact.*

The above result is a consequence of the properties of the compact future Cauchy horizon H given by the following lemmas.

Lemma 1 (Hawking and Ellis 1973 [15]) *The null geodesic generators of* H *are past complete.*

Lemma 2 (Hawking and Ellis 1973 [15]) *Let null convergence condition hold. Then the expansion* θ *and the shear* σ *of null geodesic generators of* H *are zero.*

Lemma 3 (Borde 1984 [16], Hawking 1992 [14]) *There exists an endless null geodesic generator of* H.

Lemma 4 (Beem and Królak 1998 [17]) *Let null convergence condition hold. Then all null geodesic generators of* H *are endless.*

From Lemmas 1, 2, 3 it already follows that existence of compact Cauchy horizons is incompatible with generic conditions. This is a result of Borde [16]. By Lemma 4 a compact Cauchy horizon cannot contain even one generic generator.

Thus we see that under Conditions 1 and 2 modulo certain differentiability assumptions compact Cauchy horizons are ruled out. It is interesting to note that Conditions 1 and 2 are one of the assumptions of the Hawking-Penrose singularity theorem [13].

Remark

Lemmas 1 and 3 apply to the case of a compactly generated Cauchy horizon that is not necessarily compact. The past-complete and future-endless generators of H are then contained in the compact set C in Definition 2 [14].

Let λ be a past endless achronal null geodesic. We say that λ is **past focusing** if there exists a point q on λ such that the expansion θ of the congruence of past-directed null geodesics originating from q and infinitesimally neighboring to λ becomes negative at some point p on λ. By time inverse of the above we define **future focusing** null geodesics.

Condition 3 (Strong null convergence condition) *We say that the strong null convergence condition holds if every past (future) endless achronal null geodesic terminating in the timelike ideal point of the boundary of space-time is past focusing (respectively future focusing).*

Theorem 3 (Królak 1987 [18], Królak and Rudnicki 1993 [19]) *Let S be a partial Cauchy surface with an asymptotically simple past and let the Cauchy horizon $H^+(S)$ be non-empty. If null convergence condition holds and a generator λ of $H^+(S)$ is past-focusing then the set $C := \overline{I^-(q) \cap S}$ must be compact for some point $q \in \lambda$.*

This result can be interpreted as a topological instability of Reissner-Nordström and Kerr type Cauchy horizons for which the intersection $C := \overline{I^-(q) \cap S}$ for every point q on the Cauchy horizon is non-compact. By results of Tipler [21] it follows that for Reissner-Nordström space-time an arbitrary small amount of outgoing spherically symmetric radiation in some compact neighborhood of the intersection of the event horizon with a spacelike hypersurface will cause the past strong null convergence condition to be satisfied on the Cauchy horizon. Studies of perturbations of Reissner-Nordström space-time have established not only instability of these horizons but also in a remarkably great detail the structure of the singularity of the perturbed space-time (see contribution of Brady in this volume and also recent paper by Burko [20]. The advantage of the above result is that it applies not only to a Cauchy horizon of a particular highly symmetric solution of Einstein equations but also to all Cauchy horizons of a certain topological type without any symmetries of space-time.

Definition 3 *Space-time is maximal null pseudoconvex if and only if for each compact set K there exists a compact set K' such that each maximal null geodesic segment with both endpoints in K must have its image in K'.*

Null pseudoconvexity (a condition marginally stronger than the above) together with condition of null geodesic disprisonment which follows from strong causality condition imply global solvability of inhomogeneous wave equations [22]. Thus these conditions play a similar role in the theory of partial differential equations as global hyperbolicity but are weaker. It was demonstrated that pseudoconvexity could be used in place of global hyperbolicity in study of Lorentzian geometry. For example

it implies equality of lower and upper Hausdorff limits for sequences of geodesics [22]. Intuitively, one may think of pseudoconvex space-times as those failing to have any "interior" points missing.

Theorem 4 (Beem and Królak 1992 [23]) *Let space-time be strongly causal. If both the null convergence and the strong null convergence conditions hold then space-time is maximally null pseudoconvex.*

The significance of the above theorem for cosmic censorship is that it proves a causality condition that can in some cases be used instead of global hyperbolicity to ensure predictability of space-time.

Condition 4 (Trapped surface condition) *The trapped surface condition holds if for every future-incomplete non-spacelike geodesic λ, $\lambda \in intD(S)$ where S is a regular partial Cauchy surface there exists a trapped surface $\mathcal{T}$ such that $J^+(\mathcal{T}) \cap \lambda \neq \emptyset$.*

Hawking-Penrose singularity theorem says that when space-time is strongly causal and conditions 1 and 2 hold then the existence of a trapped surface implies existence of an incomplete causal geodesic. Thus the trapped surface condition essentially says that the inverse of singularity theorem holds.

Definition 4 *Let (M, g) be a weakly asymptotically simple and empty space-time. A partial Cauchy surface S in M is said to be regular if the following conditions hold.*

1. $\overline{D^+(S, \overline{M})} \cap \lambda \neq \emptyset$ for all generators λ of $\mathcal{J}^+$.

2. S has an asymptotically simple past.

3. If $H^+(S) \neq \emptyset$ then for every past-incomplete null geodesic generator γ of $H^+(S)$ there exists a point $p \in \gamma \cap H^+(S)$ such that a set $\overline{I^-(p) \cap S}$ is compact.

The purpose of the above definition is to describe in a geometrical way what the regular initial data are i.e. to ensure that the break down of prediction does not arise from a bad choice of the initial surface. The definition originated from the concept of partial asymptotic predictability introduced by Tipler [24] that requires that at least some structure of $\mathcal{J}^+$ can be predicted from initial data. Condition 3 eliminates Cauchy horizons of the Reissner-Nordström type. If we assume that every generator of a Cauchy horizon is past-focusing then this condition follows from Theorem 3.

Theorem 5 (Królak 1986 [25], Królak and Rudnicki 1993 [26]) *Let space-time be weakly asymptotically simple and empty. If space-time contains a regular partial Cauchy surface S and if null convergence, generic, and trapped surface conditions hold then space-time is future asymptoticaly predictable from S.*

The above theorem proves predictability of space-time under the trapped surface condition which is very strong. Nevertheless the theorem is non trivial and regularity required by Definition 4 is essential for its validity.

The final conclusion of this section is that theorems based on methods of Lorentzian geometry do not restrict the occurrence of Cauchy horizons to such a degree that we can accept the cosmic censorship principle.

§3. Naked singularities in gravitational collapse

In this section we shall discuss various instances of occurrence of naked singularities in solutions of Einstein equations.

3.1. *Shell-crossing singularities*

The shell-crossing singularities are the earliest examples of naked singularities in gravitational collapse. They were first found and studied in detail in Lemaitre-Tolman-Bondi (LTB) space-times representing spherically symmetric collapse of dust [27). Even though they arise from regular initial data they were never thought to constitute a serious counterexample to cosmic censorship. One reason is that metric admits a C^0 extension through such singularities [28). Other reasons emerge during the following discussion.

3.2. *Strong curvature singularities*

Before we introduce another type of naked singularities let us recall the concept of strong curvature singularity. This is an idea of gravitationally strong singularity that destroys by crushing or stretching any object that falls into it. The idea was first defined in precise geometrical terms by Tipler [29) and then two kinds of strong curvature singularities emerged a *strong curvature singularity* that crushes all volume elements defined by Jacobi fields to zero and a *limiting focusing singularity* that causes all volume elements to decrease. It turned out those strong curvature singularities can be characterized by non-integrability of certain components of curvature tensor along causal geodesics fall into them [21), 30)]. A singularity that satisfies either strong curvature condition or limiting focusing condition is past or future focusing. A classification of strong curvature singularities in the case of spherical symmetry was given by Nolan [31).

The following conjecture emerged [29), 32)].

Conjecture 1 *If all singularities that arise in space-time are of strong curvature then cosmic censorship holds.*

3.3. *Shell-focusing singularities*

Shell focusing singularities were discovered by Eardley and Smarr [33) in spherical collapse of dust matter. They were studied in detail by Christodoulou [28) and Newman [34). These singularities have the property that they arise from the evolution of central degenerate shell of matter and they have zero mass. It was Newman [34) who has shown that shell-focusing singularities were limiting focusing singularities and consequently the Conjecture 1 put forward by Tipler and this author turned out to be false and attempts to proof cosmic censorship on that basis failed. Nevertheless by results presented in previous section, some of which originated from the attempts to prove Conjecture 1, the occurrence of naked singularities in general space-times is somewhat constrained.

Newman also showed that shell-crossing singularities do not satisfy either strong curvature or limiting focusing condition. Consequently shell-crossing singularities are gravitationally weak and integrable.

Subsequently Joshi and his school [35), 36), 37), 38)] found many examples of shell-focusing singularities that were either strong curvature or limiting focusing. Ori and Piran [39)] discovered shell-focusing singularities in spherically symmetric self-similar collapse of perfect fluid and the singularities were shown by Lake [40)] to be of strong curvature type. Lake [41)] gave the first examples of non self-similar shell-focusing singularities and Harada [42)] showed that shell-focusing singularities occur for perfect fluid with a sufficiently soft equation of state. Joshi and this author [43)] showed that strong curvature shell-focusing singularities occur in Szekeres space-times that do not have any Killing vectors.

One natural question that arises is whether generic perturbations will destroy naked singularities. Iguchi, Nakao, and Harada [44)] found that within linear perturbations odd-parity gravitational waves do not destroy the Cauchy horizon forming as a result of naked shell-focusing singularity in spherically symmetric dust collapse.

3.4. *Scalar field naked singularities*

Naked singularities were found numerically in critical collapse space-times [45)]. However, since the naked singularity is realized for a specific solution in the one parameter family, it is a subset of measure zero. Exact solutions for the case of massless scalar field containing naked singularities were discovered by Roberts [46), 47)]. Christodoulou made a complete study of gravitational collapse of scalar field. Among other things he proved [48)] that naked singularities occur in gravitational collapse of self-similar scalar field. However he showed that the *set $\mathcal{E}$ in the space of initial data leading to formation of naked singularities has positive codimension*. Thus at least for this model example we can say that cosmic censorship holds.

3.5. *Collapse of collisionless dust*

One line of thought is that in order that cosmic censorship holds the matter model must be physically realistic. Rendall [49)] proposed that an appropriate set of physically realistic set of equations is Einstein-Vlasov system. He pointed out a very appealing analogy with Poisson-Vlasov set of equations which was proven to be generically free of singularities whereas Newton's equations describing evolution of dust do exhibit singularities. Rendall pointed out that velocity dispersion present in Einstein-Vlasov case can dissolve naked singularity [49)] in the same way as in the Poisson-Vlasov case. Shapiro and Teukolsky [50)] found numerically singularities without formation of apparent horizon in axially symmetric collapse of collisionless cloud of particles. The set of equations evolved by Shapiro and Teukolsky is a special case of Einstein-Vlasov system but they assumed that particles were initially at rest. This means that in their solution there was no velocity dispersion. Recently it was shown by Harada, Iguchi and Nakao [52)] that for a spherical cloud of counterrotating particles the formation of shell-focusing singularity is prevented. As counterrotation can be interpreted as a simple model of velocity dispersion this result supports the line of attack on cosmic censorship proposed by Rendall.

Rein and Rendall proved [51)] global solvability of Einstein-Vlasov system for small initial data. One significance of this result is that in Eistein-Vlasov system there cannot be shell-crossing singularities present in LTB space-time because in that case

they can occur for arbitrarily small initial data.

§4. Observational verification of cosmic censorship

An insight into the nature of the final state of gravitational collapse can be provided by measurement of the dimensionless quantity

$$\chi := \frac{a}{m} \tag{4·1}$$

where m is mass and a is spin angular momentum per unit mass. For Kerr space-time with $\chi < 1$ weak cosmic censorship holds whereas for $\chi > 1$ it is violated. I shall discuss a number of possibilities to measure the parameter χ.

4.1. *Pulsar observations*

Observations of radio pulsars proved to be a powerful tool for studying compact objects. Pulsar in orbit with an object more compact than a neutron star is yet to be discovered. Once it is found the ratio a/m can be measured form the timing model. Two effects were considered. One is the time delay Δ_{FD} in the propagation of electromagnetic impulses from the pulsar due to frame dragging [53),54)]. The maximum of this delay is given by [55)]

$$\max(\Delta_{FD}) \approx \frac{0.8 \times 10^{-3}}{|\cos i|} \left(\frac{P_b}{1\ \mathrm{day}}\right)^{-2/3} \left(\frac{m_\odot}{10}\right)^{5/3} \chi \cos\lambda \,[\mu s], \tag{4·2}$$

where $\pi - i$ is inclination of the orbit to the line-of-sight, P_b is orbital period, $m_\odot$ is pulsar companion mass in units of solar mass, and λ is the angle between the companion spin vector and the line-of-sight. It was pointed out by Wex and Kopeikin [55)] that the measurement of the frame-dragging effect is complicated by a competing effect of the bending delay. Instead they found that the spin-induced precession will have influence on observable quantities of the timing model. In particular measurement of projected semi-major axis of the pulsar orbit and periastron advance will be perturbed by a factor proportional to χ. They have shown that one can determine parameter χ accurately within a reasonable span of observations.

4.2. *Gravitational-wave observations*

The gravitational wave signal from a binary system will carry information about the spin of the members of the system. Spin-orbit and spin-spin interactions enter respectively 1.5 PN and 2 PN corrections to the phase of the gravitational signal [56)]. From observations of gravitational-waves by laser interferometric detectors one can in principle determine masses and parameter χ of the members of the binary system [57),58)]. Simplified analysis shows that one can estimate parameter χ most accurately for a massive companion of a typical neutron star. For advanced LIGO interferometer one gets relative rms error $\frac{\Delta\chi}{\chi} \sim \frac{10}{\chi}\%$ at 200Mpc for a fiducial binary ($m_{NS} = 1.4M_\odot, m = 10M_\odot$) where m_{NS} is the mass of a neutron star. The absolute rms error in the estimation of χ is almost independent of its value and the relative error is the smaller the larger the ratio a/m.

4.3. *Measurements of X-ray binaries spectra*

The existence and properties of very compact objects like black holes can be investigated through their interaction with other bodies. For example an accretion disc can form around a compact object. The X-ray radiation emitted during accretion can be detected by satellites. Both binaries with a central object of the mass of a few tenths of solar masses and active galactic nuclei (AGN) powered by compact objects of masses equal to masses of galaxies can be sources of X-ray radiation.

In the case of stellar sized compact objects in the spectra of low mass X-ray binaries (LMXB) one observes quasiperiodic oscillations (QPO). These oscillations can be explained by relativistic effects in Kerr solutions. One possibility is that they are due to oscillations of accretion disc. The normal modes eigenfrequencies of accretion disk depend on mass and χ parameter of the spinning central object [59] and can in principle be estimated form QPO frequencies. The other possibility is that QPOs can be directly related to precession frequency of the accretion disc through Lens-Thirring effect [60].

$$\Omega_p = \frac{2\chi}{r^3}, \tag{4·3}$$

where Ω_p is precession frequency of the accretion disc and r is radius. Thus parameter χ can be determined.

In the case of supermassive objects the best tests of cosmic censorship may come from measurements of iron K_α line profiles from Seyfert type 1 galaxies. The X-ray spectra of these AGNs show evidence for line emission peaking at a rest energy of 6.4 keV. This is thought to be due to a fluorescence line from the K-shell of iron. The lines are extremely broad and have a strong asymmetry to the red. This emission is thought to originate from the innermost regions of an accretion disk around a supermassive central compact object. These lines provide the means to probe the immediate environment of a black hole. The most prominent example of such a line comes from the galaxy MCG-6-30-15 [61),62),63]. The model of the line depends on a/m of the central object and several other parameters: inner and outer radii of the accretion disc, inclination angle and emissivity index (Fabian et al. [64] and see Fanton et al. [65] for a systematic derivation). In particular the full width at zero intensity (FWZI) Δ depends on χ and we have $\Delta > 4/\sqrt{3}$ for $\chi > 1$. From current data it is not possible to conclude whether χ is 0 or 1 [66]. The recently launched CHANDRA X-ray satellite should provide more accurate measurements.

4.4. *Evolution of a/m due to accretion*

When central object accretes matter from the surrounding disc both its mass and angular momentum change. The resulting evolution of ratio a/m has been studied for the case of $a/m < 1$ by Bardeen [67] and Thorne [68] and for the case of $a/m > 1$ by de Felice [69]. Bardeen found that an initially non-rotating black hole would get spun up to extreme Kerr black hole ($a/m = 1$). Thorne [68] showed that because capture cross section is greater for negative angular momentum photons than for positive angular momentum photons black hole will be spun up to $a/m \simeq 0.998$. The case when initial a/m of the central compact object is greater than 1 was considered by de Felice [69] who showed that in this case a/m decreases. Because of

the effect studied by Thorne will also operate in this case the compact object should get spun down to $a/m \simeq 0.998$. Even though this calculation based on Kerr solution indicates (classical) instability of a naked singularity there are other physical effects like magnetic field that may change the above picture. De Felice [69] calculated that for the range $1 < \chi < 4/3\sqrt{2/3}$ the innermost stable orbits of Kerr solution have negative energy and consequently the efficiency of the photon emission can be more than 100%. Hence an extremely compact object spinning down from the initial a/m greater than 1 may be a very powerful source of energy. An early numerical work of Nakamura and Sato [70] shows that for some equations of state as a result of collapse of a rotating star with initial value of parameter $\chi > 1$ a jet forms. It was suggested by Chakravarti and Joshi [71] that naked singularities may be sources of gamma rays bursts.

It is interesting to note that all the tests of cosmic censorship discussed above do not require any new expensive instrument but they can use existing and planned observational projects. One is only required in the analysis of the data to take a sufficiently large parameter space and not assume beforehand that a/m of a compact object is necessarily less than 1.

§5. Conclusion

The results presented in our review show that there is evidence both against and for the existence of cosmic censor. It appears that analytic, numerical, and observational techniques that we have are not yet sufficiently refined to tackle this problem. Thus cosmic censorship often referred to as the most fundamental unsolved problem of general relativity remains a challenge for the next - 21st century.

§6. *Acknowledgments

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